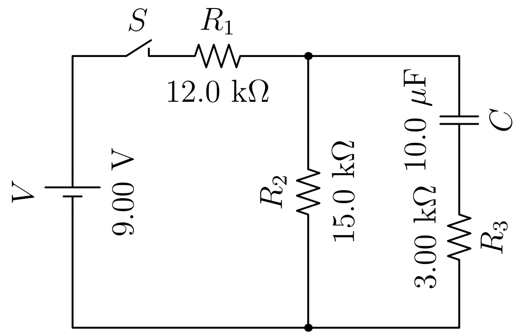
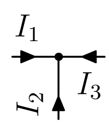
Recitation Week 7

Chapter 28

Problem 28.71. *In Figure P28.71, suppose the switch has been closed for a time interval sufficiently long for the capacitor to become fully charged. Find (a) the steady state current in each resistor and (b) the charge Q on the capacitor. (c) The switch is now opened at $t = 0$. Write an equation for the current in R_2 as a function of time and (d) find the time interval required for the charge on the capacitor to fall to one-fifth its initial value.*

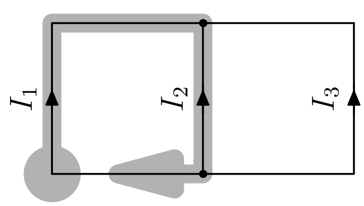


(a) Label the currents I_1, I_2, and I_3 from left to right with each current moving up in its vertical wire. In the steady state, $I_3 = 0$ (otherwise Q would be changing, which would not be "steady state"). Applying the Kirchhoff's junction rule to the top junction.



$$0 = I_1 + I_2 + I_3 = I_1 + I_2 \tag{1}$$
$$I_2 = -I_1 \ . \tag{2}$$

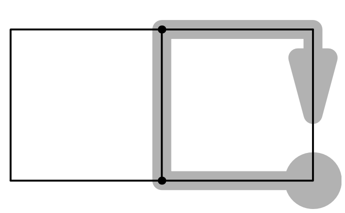
Applying Kirchhoff's loop rule to the left-hand loop moving clockwise from the lower-left corner, we have



$$0 = V - I_1 R_1 + I_2 R_2 \tag{3}$$
$$V = I_1 R_1 - I_2 R_2 = I_1 R_1 + I_1 R_2 = I_1(R_1 + R_2) \tag{4}$$
$$I_1 = \frac{V}{R_1 + R_2} = 333\ \mu\text{A} \tag{5}$$
$$I_2 = -I_1 = -333\ \mu\text{A} \ . \tag{6}$$

So we have 333 μA of current flowing clockwise through the left loop, and no current in the right loop.

(b) Applying Kirchhoff's loop rule to the right-hand loop moving clockwise from the lower-right corner, we have



$$
\begin{aligned}
0 &= -I_2R_2 - \frac{Q}{C} + I_3R_3 && (7)\\
I_2R_2 &= -\frac{Q}{C} && (8)\\
Q &= -CI_2R_2 = -(10.0\ \mu\text{F})\cdot(-333\ \mu\text{A})\cdot(15.0\ \text{k}\Omega) = 50\ \mu\text{C}\ . && (9)
\end{aligned}
$$

You could jump to the final equation for Q by using the capacitor equation $Q = CV$ and noting that the voltage drop over the capacitor must match the voltage drop $V = |I_2R_2|$ over R_2 (effectively doing Kirchhoff's loop rule in your head).

The top capacitor plate will hold the positive charge, because when the switch was first closed, the capacitor was completely discharged. The current I_1 split and flowed down through both the middle and right wire. As time went on, the charge Q built up on the capacitor, and current through the right wire slowed, stopping at equilibrium with the capacitor fully charged.

(c) With the switch opened, $I_1 = 0$. Resolving our earlier junction equation

$$
\begin{aligned}
0 &= I_1 + I_2 + I_3 = I_2 + I_3 && (10)\\
I_2 &= -I_3\ , && (11)
\end{aligned}
$$

so current will flow up through the right-hand wire ($I_3 > 0$) and down through the center wire ($I_2 < 0$) as the capacitor discharges. Repeating our Kirchhoff loop from (b),

$$
\begin{aligned}
0 &= -I_2R_2 - \frac{q}{C} + I_3R_3 && (12)\\
\frac{q}{C} &= (R_2 + R_3)I_3 = R'I_3\ , && (13)
\end{aligned}
$$

where $R' \equiv R_2 + R_3$ is shorthand to allow cleaner formulas for the rest of the problem, and q is the charge on the capacitor ($q(t = 0) = Q$).

As the capacitor discharges, $\frac{\mathrm{d}q}{\mathrm{d}t} < 0$, so $I_3 = -\frac{\mathrm{d}q}{\mathrm{d}t}$. We can then solve for $q(t)$ by integrating with respect to time.

$$
\begin{aligned}
\frac{q}{C} &= -R'\frac{\mathrm{d}q}{\mathrm{d}t} && (14)\\
\frac{-\mathrm{d}t}{R'C} &= \frac{\mathrm{d}q}{q} && (15)\\
\int_{t=0}^{t}\frac{-\mathrm{d}t}{R'C} &= \int_{t=0}^{t}\frac{\mathrm{d}q}{q} && (16)\\
\frac{-t}{R'C} &= \ln(q) - \ln(Q) = \ln\left(\frac{q}{Q}\right)\ , && (17)
\end{aligned}
$$

where $-\ln(Q)$ is a constant of integration which is determined by the conditions at $t = 0$. Raising e to either side this equation, we have

$$
\begin{aligned}
e^{\frac{-t}{R'C}} &= e^{\ln\left(\frac{q}{Q}\right)} = \frac{q}{Q} && (18)\\
q &= Qe^{\frac{-t}{R'C}} && (19)\\
I_3 &= -\frac{\mathrm{d}q}{\mathrm{d}t} = -\left(\frac{-1}{R'C}\right)Qe^{\frac{-t}{R'C}} = \frac{V_0}{R'}e^{\frac{-t}{R'C}}\ , && (20)
\end{aligned}
$$

where $V_0 = Q/C$ is the initial voltage across the capacitor. You can see that both the charge on the capacitor and the current through the loop will drop off exponentially with a time constant $R'C$ as the system discharges.

Now we can put together our knowledge of the switch-closed and switch-open cases to write an equation for I_2.

$$
I_2 = \begin{cases} \frac{-V}{R_1+R_2} = -333\ \mu\text{A} & t < 0 \\ -\frac{Q}{R'C}e^{\frac{-t}{R'C}} = \frac{CI_2(t<0)R_2}{R'C}e^{\frac{-t}{R'C}} = \frac{-VR_2}{(R_1+R_2)\cdot(R_2+R_3)}e^{\frac{-t}{R'C}} = -(278\ \mu\text{A})\cdot e^{\frac{-t}{180\ \text{ms}}} & t > 0 \end{cases} \qquad (21)
$$

(d) Plugging into our formula for $q(t)$

$$\frac{Q}{5} = q(t) = Qe^{\frac{-t}{R'C}} \tag{22}$$

$$\frac{1}{5} = e^{\frac{-t}{R'C}} \tag{23}$$

$$\ln\left(\frac{1}{5}\right) = \frac{-t}{R'C} \tag{24}$$

$$t = -R'C\ln\left(\frac{1}{5}\right) = R'C\ln(5) = 290 \text{ ms} \tag{25}$$